EXERCISE 2
OPEN PROBLEMS IN NUMBER THEORY 2017/18
DUE DATE: APRIL 12, 2018

Exercise 1. *Suppose that α is irrational of bounded type, and set $||x|| = \mathrm{dist}(x, \mathbb{Z})$. Let*

$$A(t) := \sum_{k \le t} \frac{1}{||k\alpha||}, \qquad G(K) := \sum_{k=1}^{K} \frac{1}{k||k\alpha||}$$

We saw that $A(t) \ll t \log t$. Show that

$$G(K) \ll (\log K)^2.$$

The exercises below are a review of Fourier analysis on the real line, which we will need after the break. You are exempt if you have already studied the material, otherwise these will help you get up to speed.

The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined as

$$\widehat{f}(y) := \int_{\mathbb{R}} f(x) e^{-2\pi i x y} dx$$

Exercise 2. *The convolution of "reasonable" two functions on the line (say smooth and compactly supported) is defined as*

$$f * g(x) := \int_{\mathbb{R}} f(t) g(x - t) dt$$

a) Show that convolution is commutative and associative.
b) Show that the Fourier transform of a convolution is given by

$$\widehat{f * g}(y) = \widehat{f}(y) \cdot \widehat{g}(y)$$

Exercise 3. *For $a > 0$, let $f_a(x)$ be the indicator function of the interval $[-a, a]$:*

$$f_a(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$$

a) Compute $f_a * f_b$ and plot the graph of $f_a * f_a$.
b) Find the Fourier transform $\widehat{f_a}$ and $\widehat{f_a * f_a}$.

Exercise 4. *Compute the Fourier transform of the derivative df/dx and of the translate $f(x - b)$ and dilate $f(x/T)$ in terms of the Fourier transform of f.*